Series 6.

Deadline: 26^{th} January 2015

Adress: *Korespondenční seminář iKS*
KAM MFF UK
Malostranské náměstí 25
118 00 Praha 1
Czech Republic

Problem A6. Patrik imagined a function $f\colon \mathbb{R} \to \mathbb{R}$ and noticed that it satisfies the identity

$$f(f(x) + x + y^2) = 2x + f(y)^2$$

for all $x, y \in \mathbb{R}$. Find all possible functions $f\colon \mathbb{R} \to \mathbb{R}$ that Patrik could've imagined.

Problem C6. There are $2n + 1$ participants ($n \geq 1$) in a ping-pong tournament. In the tournament, each participant plays exactly one match against every other participant. All matches take place at one table, in some order. The participants of the tournament have decided to play in such an order that, between any two matches of one participant, there are at least $n - 1$ other matches in which that participant doesn't play. Prove that one of the participants that play in the first match will also play in the last match.

Problem G6. There is an angle of magnitude α formed by a vertex A and two rays u_1, u_2 from A. Inside the angle $u_1 u_2$, a point B is given; B doesn't lie on the bisector of this angle. Furthermore, the magnitude of an angle β is given. ($\alpha < \beta < 180°$) Consider all possible pairs of points X, Y such that $X \in u_1$, $Y \in u_2$, A lies outside of the angle XBY and $|\sphericalangle XBY| = \beta$. Then, points A and B have an interesting property: from each of them, segment XY is always seen under the same angle. Prove that a third point with this property exists.

Problem N6. There's a polynomial p in n variables ($n \geq 1$) with integer coefficients. The degree of p is strictly less than n.[1] Consider all ordered n-tuples of integers $x_1, \ldots, x_n$ such that

$$p(x_1, \ldots, x_n) \equiv 0 \pmod{11} \text{ and for all } x_i,\ 1 \leq x_i \leq 11.$$

Prove that the number of such n-tuples is divisible by eleven.

[1] The degree of a polynomial p is equal to the maximum sum of exponents of its variables in some term, e.g. a polynomial $p(x, y) = x^3 y + y^2$ has degree 4.