2nd series

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Problem N2. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $m, n \in \mathbb{N}$, the identity

$$f(mn) = [m, n](f(m), f(n))$$

holds, where $[x, y]$ and (x, y) denote the least common multiple and greatest common divisor of x, y, respectively.

Problem C2. Patrik has been playing with square grids, where each grid cell was either white or black. He realised that grids seem more beautiful to him than some other ones and also that some of them seem more interesting to him than other ones. Let's call an $m \times n$ grid *beautiful*, if for any two of its rows, there's at most one column such that the rows both have a black cell in that column. Next, let's call a grid *interesting*, if it is beautiful, but if we colour any white square black, it will stop being beautiful. What's the least number of black squares that can exist in an interesting grid?

Problem G2. Consider a triangle ABC with incenter I. Let's denote by D, E, F the points of contact of its incircle with sides BC, AC, AB. Next, let's denote by k, l the circles inscribed in quadrilaterals $BDIF$, $CEID$. Prove that one of the common tangents of circles k, l passes through point A.

Problem A2. Let $x_1 \leq x_2 \leq \ldots \leq x_n, y_1 \geq y_2 \geq \ldots \geq y_n$ be real numbers satisfying the equality

$$\sum_{i=1}^{n} i x_i = \sum_{i=1}^{n} i y_i \,.$$

Prove that for any $\alpha \in \mathbb{R}$,

$$\sum_{i=1}^{n} x_i [i\alpha] \geq \sum_{i=1}^{n} y_i [i\alpha]$$

holds, where $[x]$ denotes the greatest integer not exceeding x.