5th series

Date: 21st January 2013

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Problem G5. Let ABC be a triangle inscribed in a circle ω and P a variable point on the arc BC of ω not containing vertex A. Denote by I, J the incenters of triangles PAB, PAC, respectively. Prove that as P varies,

(a) the circles with diameters IJ all have a common point,

(b) the midpoints of the segments IJ all lie on a single circle,

(c) the circumcircles of the triangles PIJ all have a common point.

Problem A5. Let $n \geq 2$. Given that $x_1, \ldots, x_n$ are positive real numbers satisfying $x_1^2 + \cdots + x_n^2 = 1$, determine (in terms of n) the minimal possible value of the expression

$$\sum_{j=1}^{n} \frac{x_j^5}{-x_j + \sum_{k=1}^{n} x_k}.$$

Problem N5. Let $n \geq 2$ and let $P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + 1$ be a polynomial with positive integer coefficients such that $a_k = a_{n-k}$ for $k = 1, \ldots, n-1$. Prove that there exist infinitely many pairs (x, y) of positive integers such that both $x \mid P(y)$ and $y \mid P(x)$.

Problem C5. Czechoslovakia is a country with at least one village in which some pairs of villages are connected by roads. Mirek the baker founded his M-Bakery company and wants to build stores in several villages (at most one store per village) in such a way that every citizen of Czechoslovakia who can't buy M-croissants in their own village can buy them in one of the neighbouring ones. Prove that the number of ways to do so is odd.