TRiKS ko

Úloha 1. *An acute isosceles triangle ABC ($AB = AC$) is inscribed in a circle with center O. Rays BO and CO intersect the sides AC and AB at B' and C', respectively. A straight line ℓ parallel to AC is drawn through C'. Prove that ℓ is tangent to the circumcircle of triangle $B'OC$.*

Řešení. Let ℓ and AO meet at T. By symmetry and parallel lines we have $\angle B'TO = \angle C'TO = \angle OAC = \angle ACO = \varphi$, hence T lies on the circumcircle of triangle $B'OC$. Similarly, $\angle OB'T = \angle OC'T\angle ACO = \varphi$, hence $\angle OB'T = \angle C'TO$ as desired. □

Úloha 2. *Each cell of a 100×100 board is painted either black or white such that all the cells adjacent to the border of the board are black. It turned out that no 2×2 square of the board is one-colored. Prove that there exists a 2×2 square with two diagonally touching black squares and two diagonally touching white squares.*

Řešení. Assume otherwise and consider segments of the grid that separate two squares of different colors. Call such segments "separators". It's easily checked that by assumption, each 2×2 square contains 2 separators. Moreover, each separator is contained in precisely two 2×2 squares (no separators are adjacent to the boundary). Thereofore the number of separators is the same as the number of 2×2 squares, that is 99^2. On the other hand, each row and column contains an even number of separators, a contradiction. □

Úloha 3. *Initially, a positive integer n is written on the board. At any moment, Misha can choose any number $a > 1$ on the board, erase it, and write on the board all the divisors of a, except for a itself (the same number can appear multiple times on the board). After a while it turned out that n^2 numbers were written on the board. Find all n for which this could have happened.*

Řešení. The only solution is $n = 1$. Recall that

$$\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} < 1$$

since $1/(t(t+1)) = 1/t - 1/(t+1)$.

We proceed by induction. The base case is trivial. Let $1 = d_1 < d_2 < \cdots < d_{k+1} = n$ be all the divisors of n. Erase n and replace it by all its divisors. By induction, there can't be more than $d_1^2 + d_2^2 + \cdots + d_k^2$ numbers on the board. Note that if t is a divisor of n then so is n/t. Hence we can rewrite

$$\begin{aligned} d_1^2 + d_2^2 + \cdots + d_k^2 = \frac{n^2}{d_{k+1}^2} + \frac{n^2}{d_k^2} + \cdots + \frac{n^2}{d_2^2} \leq \\ \leq n^2 \left(\frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \right) < n^2 \cdot 1 = n^2 \end{aligned}$$

and conclude. $\square$