Series 5.

Deadline: December the 9th, 2013

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Problem A5. Prove the equality

$$\lfloor \sqrt[2]{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor + \cdots + \lfloor \sqrt[n]{n} \rfloor = \lfloor \log_2(n) \rfloor + \lfloor \log_3(n) \rfloor + \cdots + \lfloor \log_n(n) \rfloor .$$

for all integers $n \geq 2$.

Problem G5.

We're given an acute triangle ABC. We call a circle k *clever*, if it passes through the vertex A, intersects sides AB and AC (we denote the intersection points X_k, Y_k in this order) and the intersection points of line segments BY_k and CX_k lies on k. Prove that all clever circles pass through a fixed point other than A.

Problem C5. There are 2 white (and no black) tokens lying on a circle. We're allowed to perform the following operations:

- Add a white token on the circle and invert the colors of its neighboring tokens - a white token turns black and a black one turns white.
- If there are at least 3 tokens left on the circle, remove a white token and invert the colors of its neigboring tokens.

Is it possible to reach a state, in which there are two black tokens (and no white ones) left on the circle?

Problem N5. David was analyzing a monic polynomial[1] p with integer coefficients. He was trying to prove that p has no integer roots, by trying to find a positive integer n such that for all $k \in \{0, \ldots, n-1\}$, the condition

$$p(k) \not\equiv 0 \pmod{n}$$

would hold. However, he found out that no such n exists. Must the polynomial p have an integer root, then?

[1] A monic polynomial has the leading coefficient equal to 1, for example a polynomial $x^3 + 2x^2 + 3$ is monic, while $2x^2 + 1$ is not.